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SYNTHESIS OF THE OPTIMAL CONTROL FOR A LINEAR SYSTEM WITH TWO PHASE CONSTRAINTS*

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The synthesis of a control for a system described by a linear, second-order differential equation with constant coefficients (an oscillatory section) and when two constraints are imposed on the phase coordinates (one of them mixed) is given. The properties of the optimal phase trajectories are described.

1. Formulation of the problem. The following problem arises when constructing the servos for measuring systems. The measuring system intended for tracking an external object is initially given the angular elevation of the object. On receiving the signal, the servo of the system turns its sighting beam in the prescribed direciton. The fastest possible rate of sweep of the sighting beam must be ensured, taking into account the restriction imposed on its rate of motion and on the maximum power demand allowed.

Using this formulation, we will separate the problem of synthesizing the optimal response control $\bar{u}^\circ$ transforming the system

$$dy/dt = -2\xi/Ty - \varphi/T^2 + \bar{u}/T^2, \quad d\varphi/dt = y \tag{1.1}$$
$$T > 0, \quad 0 < \xi < 1$$

from the arbitrary admissible points φ, y to the origin of coordinates, with the following constraints imposed on the control $\bar{u}$ and phase coordinates:

$$|\bar{u}| \leqslant \bar{u}_0, \quad |y| \leqslant 2\bar{y}_0, \quad |y dy/dt| \leqslant 4\bar{P}_0 \tag{1.2}$$

(the second condition describes the velocity constraint and the third the power constraint).

Next we consider the case when $\xi \notin (0.519 \div \sqrt{2}/2)$ (see Sect.5).

We know /1/ that the form of the optimal trajectories sought depends, under the constraints given in (1.2), mainly on the form of the roots of the characteristic equation (1.1)

$$\lambda_{1,2} = \varkappa \pm \mu i, \quad \varkappa = -\xi/T < 0, \quad \mu = (1-\xi^2)^{1/2}/T$$

Transforming the variables

$$\begin{Vmatrix} z \\ \delta \end{Vmatrix} = \begin{Vmatrix} 1/2 & 0 \\ 1/2\varkappa\mu^{-1} & -A \end{Vmatrix} \begin{Vmatrix} y \\ \varphi \end{Vmatrix}; \quad \left(A = \frac{\varkappa^2+\mu^2}{2\mu}\right) \tag{1.3}$$

we reduce system (1.2) and the constraints (1.2) to a form suitable for our investigation

$$dz/dt = \varkappa z + \mu\delta + \mu u, \quad d\delta/dt = -\mu z + \varkappa\delta + \varkappa u \tag{1.4}$$
$$|u| \leqslant u_0, \qquad u_0 = A\bar{u}_0 \tag{1.5}$$
$$|z| \leqslant y_0, \qquad y_0 = \bar{y}_0/2 \tag{1.6}$$

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$$z\,(\varkappa z + \mu\delta + \mu u) \leqslant P_0 \tag{1.7}$$

$$-P_0 \leqslant z\,(\varkappa z + \mu\delta + \mu u), \quad P_0 = \bar{P}_0/4 \tag{1.8}$$

The main difference between problem (1.4)–(1.8) and that studied in /2/, is the presence of the mixed phase constraints (1.7), (1.8) and the description of system (1.4) in terms of the oscillatory section.

2. Conditions of optimality of the control u° [1, 3, 4]. There is a constant $\alpha_J \geqslant 0$, and functions $\psi_1, \psi_2, \psi_t, d\mu_1\pm/dt, d\mu_2\pm/dt$ satisfying the conditions

$$\frac{d\mu_1^\pm}{dt}(\pm z - y_0) = 0, \quad \frac{d\mu_1^\pm}{dt} \geqslant 0$$

$$\frac{d\mu_2^\pm}{dt}[\pm z\,(\varkappa z + \mu\delta + \mu u) - P_0] = 0, \quad \frac{d\mu_2^\pm}{dt} \geqslant 0$$

(the plus (minus) sign corresponds to the case when $z = y_0\,(z = -y_0)$, and relations (1.7), (1.8)) and the conjugate system of differential equations (t_1 is the instant of termination of the control)

$$-\frac{d\psi_1}{dt} = \frac{\partial \mathbf{H}}{\partial z} = \varkappa\psi_1 - \mu\psi_2 - (\pm 1)\frac{d\mu_1^\pm}{dt} - \frac{d\mu_2^\pm}{dt}[\pm(\varkappa z + \mu\delta + \mu u) \pm \varkappa z] \tag{2.1}$$

$$-\frac{d\psi_2}{dt} = \frac{\partial \mathbf{H}}{\partial \delta} = \mu\psi_1 + \varkappa\psi_2 - \frac{d\mu_2^\pm}{dt}(\pm\mu z)$$

$$-\frac{d\psi_t}{dt} = \frac{\partial \mathbf{H}}{\partial t}, \quad \psi_t(t_1) = -\alpha_J$$

$$\mathbf{H} = \psi_1(\varkappa z + \mu\delta + \mu u) + \psi_2(-\mu z + \varkappa\delta + \varkappa u) + \psi_t - \frac{d\mu_1^\pm}{dt}(\pm z - y_0) - \frac{d\mu_2^\pm}{dt}[\pm z\,(\varkappa z + \mu\delta + \mu u) - P_0]$$

The optimal control $u^\circ(t)$ ensures that at every instant t

$$\max_{u \in V^*_{z,\delta,t}} [\psi_1(\varkappa z + \mu\delta + \mu u) + \psi_2(-\mu z + \varkappa\delta + \varkappa u)] \tag{2.2}$$

$$V^*_{z,\delta,t} = \{u : |z\,(\varkappa z + \mu\delta + \mu u)| - P_0 \leqslant 0, \ |u| \leqslant u_0\}$$

on the optimal phase trajectory $z^\circ(t), \delta^\circ(t)$.

3. Controlling the system within the phase constraints. Let us consider the motion of system (1.4) over the time intervals in which the system does not violate the constraints (1.6), (1.7). Then from Sect.2 it follows that

$$d\mu_1\pm/dt \equiv 0, \quad d\mu_2\pm/dt \equiv 0$$

and we arrive at the problem studied in /1/.

From the solution of (2.1) we have

$$\psi_1(t) = Ce^{-\varkappa t}\sin(\mu t + \alpha), \quad \psi_2(t) = Ce^{-\varkappa t}\cos(\mu t + \alpha) \tag{3.1}$$

(C, α are arbitrary integration constants).

The optimal control

$$u^\circ = u_0 \operatorname{sign}(\mu\psi_1 + \varkappa\psi_2) \tag{3.2}$$

retains its sign over the maximum time interval $\Delta t = \pi/\mu$.

We note that within the phase constraints we have

$$d(\mu\psi_1 + \varkappa\psi_2)/dt = -2\varkappa(\mu\psi_1 + \varkappa\psi_2) + (\mu^2 + \varkappa^2)\psi_2 \tag{3.3}$$

Let us denote by $e_\pm = (a_\pm^{(1)}, a_\pm^{(2)})$ the stationary points of system (1.4) when $u = \pm u_0$. We have

$$\varkappa a_\pm^{(2)} + \mu a_\pm^{(1)} + \mu(\pm u_0) = 0$$

$$-\mu a_\pm^{(2)} + \varkappa a_\pm^{(1)} + \varkappa(\pm u_0) = 0$$

$$a_\pm^{(2)} = 0, \quad a_\pm^{(1)} = -(\pm u_0)$$

and relative to these points we have

$$dz/dt = \varkappa z + \mu(\delta - a_\pm^{(1)}) \tag{3.4}$$

$$d(\delta - a_\pm^{(1)})/dt = -\mu z + \varkappa(\delta - a_\pm^{(1)})$$

or

$$z = Ce^{\varkappa t}\sin(\mu t + \beta), \quad \delta - a_\pm^{(1)} = Ce^{\varkappa t}\cos(\mu t + \beta) \tag{3.5}$$

(C, β are arbitrary integration constants).

The segments of the phase trajectories of system (1.4) with $\pm u_0$, passing through the origin of coordinates, form the end segments of the optimal phase trajectories of the system along which the system enters the origin of coordinates directly. The optimal control changes its sign on the same segments. We will call them the first segments of the switch-over line SL. The beginning and end of the first segment of SL lie on the straight line $z = 0$. The second, adjacent segment, etc. of SL are constructed according to the rules given in /1/. SL with the first segment (1) and second segment (2) is shown in Fig.1. The circles show the values of the optimal control in the corresponding regions of the phase space and (δ, z) is the synthesis of optimal control within the phase constraints.

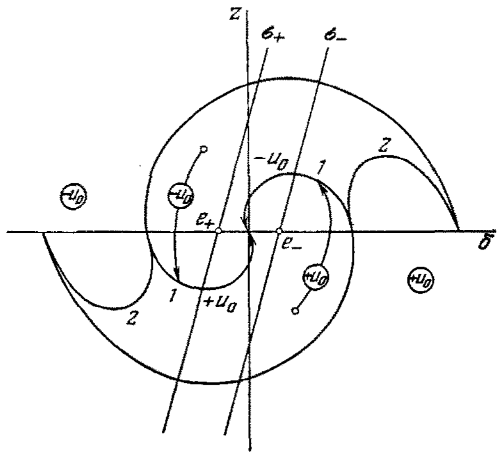

Fig.1

We note that the extremal values of the z coordinate at the branches of the optimal trajectories with $u^\circ = \pm u_0$ lie on the straight lines

$$z = \mu\,(\delta - a_{\pm}^{(1)})/\varkappa \tag{3.6}$$

(the lines σ_+, σ_- in Figs.1—6).

4. Controlling the system at the boundary of the velocity phase constraints. When the system moves along the constraint boundary (1.6), it cannot emerge at the constraints (1.7), (1.8) (since $dz/dt = 0$). Then $d\mu_2\pm/dt = 0$ and from (1.4) we have

$$u^\circ = (\varkappa\,(\pm y_0) + \mu\delta)/\mu \tag{4.1}$$

which implies that under these conditions (see (2.2), (2.1)) the following relations hold:

$$\mu\psi_1 + \varkappa\psi_2 \equiv 0, \quad d\psi_2/dt \equiv 0$$
$$d\psi_1/dt \equiv 0 = (\varkappa^2 + \mu^2)\,\psi_2/\mu + (\pm 1)\,d\mu_1\pm/dt$$

Since $d\mu_1\pm/dt \geqslant 0$, it follows that on reaching $z = y_0$ we must have $\psi_2 \leqslant 0$, and on reaching $z = -y_0$ we must have $\psi_2 \geqslant 0$.

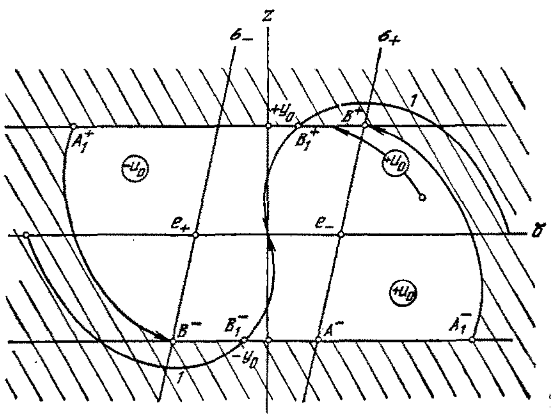

Fig.2

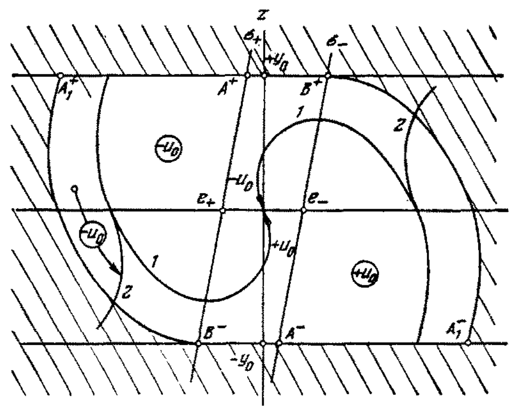

Fig.3

Let us consider reaching $z = y_0$ $(z = -y_0)$. From (1.4) and (4.1) we have

$$d\delta/dt = -(\pm y_0)(\mu^2 + \varkappa^2)/\mu$$

During the motion along the boundary the coordinate δ decreases (increases) monotonically. The range of its variation is determined by the constraints imposed on the control

$$-u_0 \leqslant [\varkappa\,(\pm y_0) + \mu\delta]/\mu \leqslant u_0$$

The boundary points of the admissible range of motions of the system within the phase constraint $z = \pm y_0$ are denoted in Figs.2, 3 by $\mathbf{A}\pm, \mathbf{B}\pm$. They represent the points of intersection of the straight lines (3.6) with $z = \pm y_0$.

Let us consider the conditions under which the points δ, z, lying within the phase constraints of the problem (1.4)—(1.8), from their region reach the boundary.

This is possible on the segments $\mathbf{B}^+, \mathbf{B}_1^+$ and $\mathbf{B}^-, \mathbf{B}_1^-$ when the first segment of SL intersects the phase boundary (Fig.2), and at the points $\mathbf{B}^+, \mathbf{B}^-$ when it does not intersect it (Fig.3). The departure from the phase boundary into the region of the points δ, z, lying within the phase boundaries of the problem (1.4)–(1.8) takes place from the point of the boundary A_1^+, B_1^+ and B_1^-, A_1^- for the case shown in Fig.2, and from the points A_1^+, B^+, A^-_1, B^- of the boundary for the case shown in Fig.3.

The region of admissible initial conditions of the system from which it can be transported to the origin of coordinates without violating condition (1.6) and without taking into account constraints (1.7) and (1.8), is shown in Figs.2, 3 without hatching (the remaining notation follows that of Fig.1). In general, the boundaries shown in Fig.3 may also contain several segments of SL.

5. Controlling the system at the boundary of the phase power constraint.

We see at once that we cannot have simultaneous motion along the boundaries (1.7), (1.8) and (1.6), i.e. $d\mu_1\pm/dt = 0$.

From the motion along the boundary (1.7), (1.8) we must have

$$u^\circ = \pm P_0/\mu z - \varkappa z/\mu - \delta \tag{5.1}$$

(the plus (minus) sign refers to the motion along the boundary (1.7), (1.8)). The phase trajectory of the system satisfies the equations

$$dz/dt = \pm P_0/z, \quad d\delta/dt = \pm \varkappa P_0/\mu z - z(\varkappa^2 + \mu^2)/\mu \tag{5.2}$$

From (5.2) we obtain directly (C_δ is the integration constant)

$$\delta(z) = \varkappa z/\mu - (\pm 1)\, z^3 (\mu^2 + \varkappa^2)/(3\mu P_0) + C_\delta \tag{5.3}$$

In the motion along (1.7) (see (5.2)) the coordinate z (see Fig.4) increases (decreases) monotonically in the upper (lower) half-plane $z > 0$ $(z < 0)$, and thecoordinate δ decreases (increases) monotonically. From (5.1) and (5.3) we find that in the upper (lower) half-plane

$$d^2u^\circ/dz^2 > 0 \quad (d^2u^\circ/dz^2 < 0)$$

and the control attains its minimum (maximum) at the unique stationary point $(du^\circ/dz = 0)$ of the function $u^\circ(z)$

$$z^+_{\substack{min\\max}} = \pm [P_0(\varkappa + (2\varkappa^2 + \mu^2)^{1/2})/(\mu^2 + \varkappa^2)]^{1/2} \tag{5.4}$$

(the plus and minus signs preceding the square brackets refer to the upper and lower half-plane respectively). This extremal value must belong to $[-u_0, +u_0]$, therefore the straight line $z = \overset{+}{z}_{\min}$ $(z = \overset{+}{z}_{\max})$ can be intersected by the phase trajectories only on the segment

$$\frac{P_0}{\mu z^+_{\substack{min\\max}}} - \frac{\varkappa}{\mu} z^+_{\substack{min\\max}} - u_0 \leqslant \delta \leqslant -u_0 + \frac{P_0}{\mu z^+_{\substack{min\\max}}} - \frac{\varkappa}{\mu} z^+_{\substack{min\\max}}$$

Fig.4 shows such limit trajectories of the system (5.2). The dashed line denotes the trajectory (5.3) which moves completely within (1.7) except for a single point. The set of points satisfying the condition

$$z(\varkappa z + \mu\delta + \mu u) = P_0, \quad |u| \leqslant u_0$$

(regions A_2, B_2 in Fig.4) represents a region in which the system can move, with help of the control, along the boundary (1.7). A motion in which condition (1.7) is satisfied is impossible within the region A_2^-, B_2^- (Fig.4), sinc the phase constraint (1.7) will be violated here. The region A_2^- is determined by the constraint imposed on the control from below, and the region B_2^- from above.

When the motion (see (5.2) takes place along (1.8), the coordinate z decreases (increases) in the upper (lower) half-plane $z > 0$ $(z < 0)$ and the coordinate δ changes the sign of its rate of change on the straight lines (Fig.5)

$$z_\delta = \pm[-\varkappa P_0/(\mu^2 + \varkappa^2)]^{1/2} \tag{5.5}$$

In the upper (lower) half-plane we have

$$d^2u^\circ/dz^2 < 0 \quad (d^2u^\circ/dz^2 > 0)$$

and at the unique stationary point $du^\circ/dz = 0$)

$$z^-_{\substack{max\\min}} = \pm [-P_0(\varkappa - (2\varkappa^2 + \mu^2)^{1/2}]/(\mu^2 + \varkappa^2)]^{1/2} \tag{5.6}$$

(where the plus and minus signs preceding the square brackets refer to the upper and lower half-plane respectively), the control u attains its maximum (minimum). We find that the straight line $z = \bar{z}_{\max}$ $(z = \bar{z}_{\min})$ can be intersected by the phase trajectories only on the segment

$$-\frac{P_0}{\mu z^-_{\substack{max\\min}}} - \frac{\varkappa}{\mu} z^-_{\substack{max\\min}} - u_0 \leqslant \delta \leqslant u_0 - \frac{P_0}{\mu z^-_{\substack{max\\min}}} - \frac{\varkappa}{\mu} z^-_{\substack{max\\min}}$$

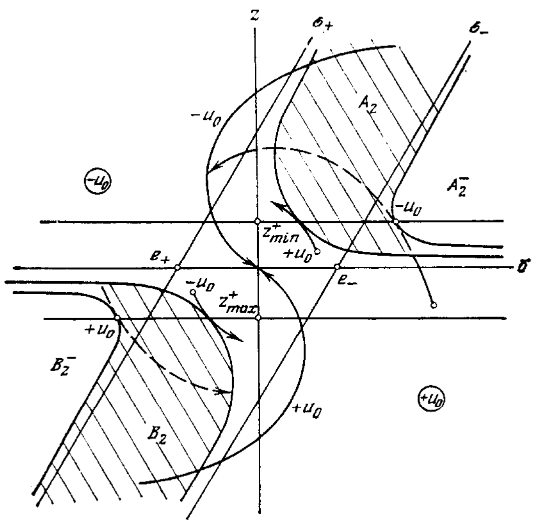

Fig.4

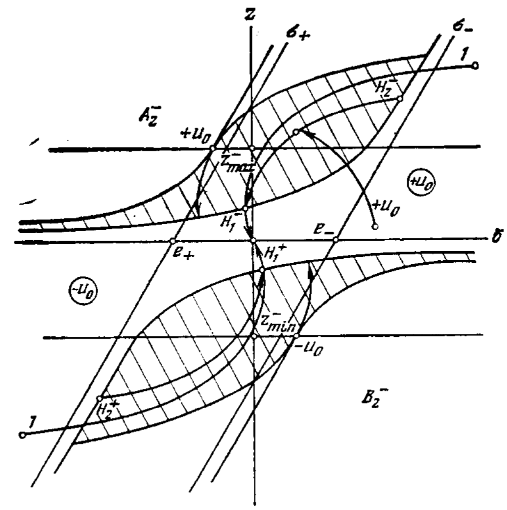

Fig.5

By analogy with the previous exercises, Fig.5 shows that $\mathbf{A}_2$, $\mathbf{B}_2$ regions in which the system can move, under the control (5.1), along the boundary (1.8) of the phase constraint, $\mathbf{A}_2^-$, $\mathbf{B}_2^-$ are the regions in which the constraint is violated, with $\mathbf{A}_2^-$ defined by the constraints imposed on the control from above, and $\mathbf{B}_2^-$ by hose from below.

In the subregion of $\mathbf{A}_2$ and $\mathbf{B}_2$ (Figs.4, 5) hatched with oblique lines, the system, having emerged at the boundary (1.8), moves under the control (5.1) along the boundary without crossing it, and leaves the constraint after some time has elapsed. Under the constraints imposed on ξ the system does not leave the hatched region.

Direct comparison of $d\delta/dz$ on the trajectories passing along the boundaries of phase constraints (1.7), (1.8) and on the trajectories emerging at the boundaries from the region bounded by the phase constraints, shows that at the instant of arrival and departure the trajectories touch each other. A kink is observed when the trajectories pass from (1.7) to (1.8).

We note that points with $d\delta/dz = 0$ on the optimal trajectories within the phase constraints lie on the straight lines

$$-\mu z + \varkappa(\delta - u_0) = 0, \quad z > 0$$

$$-\mu z + \varkappa(\delta + u_0) = 0, \quad z < 0$$

which intersect the boundary (1.8) at the points with coordinate

$$z_\delta = \pm[-\varkappa P_0/(\varkappa^2 + \mu^2)]^{1/2}$$

6. Synthesis of optimal control. Investigation of the properties of the optimal control carried out in Sect.3–5, enables us to construct the synthesis of the optimal control, i.e. to obtain the dependence of u° on the running values of the phase coordinates z, δ of the system.

Indeed, if at any instant t the phase point is found within the boundaries of the phase constraint $|z| \leqslant y_0$ and $V^*_{z,\delta,t} = \{|u| \leqslant u_0\}$, the quantity $u^\circ = \pm u_0$ and the optimal motion are governed for such t by the laws established in Sect.3. When the phase point emerges at some instant t at the boundary (1.6) or (1.7), (1.8), then the optimal motion must be subject to the laws established in Sect.4 or 5 respectively. The optimal instant of departure of the phase point from these boundaries (and the phase point cannot be found simultaneously at the boundaries (1.6) and (1.7), (1.8)), is determined uniquely by the fact that, after the system has departed from the boundaries (1.6) and (1.7), (1.8), the optimal control takes only the limit values $\pm u_0$ and does not change its sign before reaching the origin of coordinates.

We note that two, essentially different forms of the system exist: SL intersects the boundary (1.8) (see Fig.6), and SL does not intersect the boundary (1.8).

Let us construct the control for systems of the first type, as the most complex ones, and let the mutual distribution of the contraints be such that it is possible to reach the boundary (1.6) and on (1.7), (1.8) (see Fig.6).

Let the initial position of the phase point be strictly within the constraints, i.e. $|z| \leqslant y_0$, $V^*_{z,\delta,t} = \{|u| \leqslant u_0\}$ (let us say, to be specific, that it lies on the δ axis and within the first segment of SL). Since the pattern of the phase plane and the points $1^\pm - 4^\pm$ are symmetrical about zero, Fig.6 shows only the trajectories of motion of the system for the points $1^+ - 4^+$.

When the system moves from the initial points of type $1^{\pm}$, it does not reach the boundaries (1.6)–(1.8) (this case was studied in Sect.3). Moving under the control $\pm u_0$ (the principle of choosing the sign is clearly shown in Fig.6), the system reaches SL and moves along it to arrive at the origin of coordinates.

When the system moves from points of the type $2^{\pm}$ (the case was studied in Sect.3,5), it proceeds under the control $\pm u_0$ to the boundary (1.7). After this it moves along it under the control (5.1) and $\mathbf{P}_0 = \pm\mathbf{P}_0$ to emerge at the boundary (1.8) on the line $\mathbf{H}_1^{\pm}, \mathbf{H}_3^{\pm}$ (the velocity constraints are not attained) and $\mathbf{H}_1^{\pm}$ is the point of intersection of the boundary (1.8) with the finite segment of SL, while $\mathbf{H}_3^{\pm}$ is the point of intersection of the boundary (1.6) with the trajectory of motion of the system along the boundary (1.8), passing through the point $\mathbf{H}_1^{\pm}$. Moving along $\mathbf{H}_1^{\pm}\mathbf{H}_3^{\pm}$ to the point $\mathbf{H}_1^{\pm}$ under the control (5.1) with $P_0 = -P_0$ (the control attains the value $\pm u_0$ at the point $\mathbf{H}_1^{\pm}$), the system emerges at the point $\mathbf{H}_1^{\pm}$ at the end segment of SL and proceeds along it to the origin of coordinates.

The optimal motion from the initial points of type $3^{\pm}$ differs from the optimal motion from type $2^{\pm}$ points in that after reaching the boundary (1.7) and moving along it for some time (here the control is constructed according to the rules of Sect.5), the system reaches the

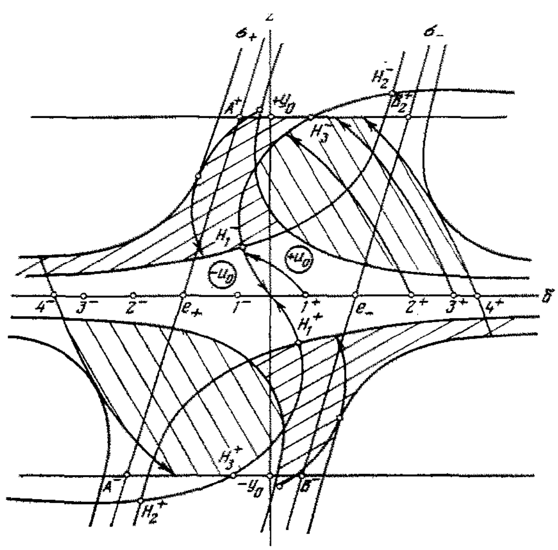

Fig.6

phase boundary (1.6) (the optimal control on it is constructed according to the rules of Sect. 4). Here this boundary cannot be reached outside the segment $\mathbf{A}^+\mathbf{B}^+(\mathbf{A}^-\mathbf{B}^-)$. The system moves along the phase boundary towards the point $\mathbf{H}_3^{\pm}$. When the point $\mathbf{H}_3^{\pm}$ is reached, a switchover takes place to the control ensuring that the system moves along the trajectory $\mathbf{H}_3^{\pm}\mathbf{H}_1^{\pm}$ (the optimal control is constructed here according to the rules of Sect.5). The subsequent motion of the system is identical with that discussed above.

The point $4^{\pm}$ is a limit point of the set of type $3^{\pm}$ points. The motion of all points to the right $\delta > 0$, and left $\delta < 0$ of it leads to violation of the power constraint (1.7). The motion from this point follows the rules given for the optimal motions from the initial type $3^{\pm}$ points.

The problem was proposed by V.G. Razumov and V.V. Slatin.

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